Recurring Digits in the Ackermann Function

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Abstract

On average, each successive tetrate of an integer gains an additional reccuring digit, begining with the least significant digit. This occurs regardless of the base of the digits. Reccuring digits occur in all higher arithmatic operators of the Ackermann function beyond exponentiation.

1 Tetrates of 3

Even with symbolic software, tetration or iterated exponentiation typically produces numbers to large to manipulate. One method to circumvent this limitation is to work with tetration using modulo arithmetic. Let $\alpha_0 = 0,\ \alpha_n = 3^{\alpha_{n-1}}$, then $\alpha_1 = 1,\ \alpha_2 = 3,\ \alpha_3 = 27,\ \alpha_4 = 7625597484987$, and in general $\alpha_{n+1} = {}^n3$.

$$\begin{aligned}
{}^03 &\equiv 1 \\
{}^13 &\equiv 3 \\
{}^23 &\equiv 27 \\
{}^33 &\equiv 97484987 \\
{}^43 &\equiv 739387 \\
{}^53 &\equiv 60355387 \\
{}^63 &\equiv 26595387 \\
{}^73 &\equiv 195387 \\
{}^83 &\equiv 4195387 \\
{}^93 &\equiv 64195387 \\
{}^{10}3 &\equiv 64195387
\end{aligned}$$

Table 1: Tetrates of 3 $(mod\ 10^8)$

In the tetrates of 3, the the kth least significant digit "freezes", recurring in all ${}^n3$ where $n \geq k$. Consider how $3^{387} = 3^{300} \cdot 3^{80} \cdot 3^7$. Several identities of the Euler Phi function $\varphi(n)$ will be helpful in showing how congruences occur with $1\ (mod\ 10^k)$ for higher powers of 3^{10^n}.

Let a and b be two positive integers that are coprime, they have no common divisors, $\gcd(a, b) = 1$,then

$$a^{\varphi(b)} \equiv 1\ (mod\ b)$$

For $m > 1$,

$$\varphi(b^m) = b^{m-1}\varphi(b)$$

When working in a base b,

$$a^{\varphi(b^m)} = a^{b^{m-1}\varphi(b)} = (a^{\varphi(b)})^{b^{m-1}} \equiv 1 \ (mod \ b^m)$$

Note how $\varphi(100) = 40, \varphi(1000) = 400, \varphi(10000) = 4000$. While the Euler Phi function assures that if $m = \varphi(n)$ then $a^m \equiv 1 \ (mod \ n)$, the Carmicheal function $\lambda(n)$ gives the smallest value of $m = \lambda(n)$ such that $a^m = a^{\lambda(n)} \equiv 1 \ (mod \ n)$ The Carmicheal function gives values of $\lambda(100) = 20, \lambda(1000) = 100, \lambda(10000) = 500$.

$$3^7 = 2187 \equiv 87 \ (mod \ 10^2)$$

$$3^{\lambda(10^2)} = 3^{20} = 3486784401 \equiv 1 \ (mod \ 10^2)$$

$$3^{80} = (3^{20})^4 \equiv 1^4 \equiv 1 \ (mod \ 10^2)$$

$$3^{\lambda(10^3)} = 3^{100} \equiv 1 \ (mod \ 10^3)$$

$$3^{300} = (3^{100})^3 \equiv 1^3 \equiv 1 \ (mod \ 10^3)$$

$$3^{300} \cdot 3^{80} = 3^{380} \equiv 1 \ (mod \ 10^2)$$

Therefore $3^7 = 2187$ determines the last two digits of 3^{387}.

$$3^{387} = 3^{380} \cdot 3^7 \equiv 1 \cdot 3^7 \equiv 87 \ (mod \ 10^2)$$

$3^{10^0} \equiv$	3	$7^{10^0} \equiv$	7
$3^{10^1} \equiv$	59049	$7^{10^1} \equiv$	282475249
$3^{10^2} \equiv$	621272702107522001	$7^{10^2} \equiv$	691459636928060001
$3^{10^3} \equiv$	102768902855220001	$7^{10^3} \equiv$	141207731280600001
$3^{10^4} \equiv$	498105206552200001	$7^{10^4} \equiv$	549213512806000001
$3^{10^5} \equiv$	250669865522000001	$7^{10^5} \equiv$	205755128060000001
$3^{10^6} \equiv$	468478655220000001	$7^{10^6} \equiv$	419551280600000001
$3^{10^7} \equiv$	862786552200000001	$7^{10^7} \equiv$	395512806000000001

Table 2: Double powers of n in 3^{10^n} and $7^{10^n} (mod \ 10^{18})$

The Table 2 shows how the powers of ten in these congruences grow arbitrarily large in the cases of 3 and 7. This is an direct consequence of both 3

$2^{3^0} \equiv$	2_3	$3^{2^0} \equiv$	11_2
$2^{3^1} \equiv$	22_3	$3^{2^1} \equiv$	1001_2
$2^{3^2} \equiv$	200222_3	$3^{2^2} \equiv$	1010001_2
$2^{3^3} \equiv$	100100112222002222_3	$3^{2^3} \equiv$	1100110100001_2
$2^{3^4} \equiv$	22000021220022222_3	$3^{2^4} \equiv$	11010111010000001_2
$2^{3^5} \equiv$	122020112200222222_3	$3^{2^5} \equiv$	100011111010000001_2
$2^{3^6} \equiv$	22121122002222222_3	$3^{2^6} \equiv$	101011110100000001_2
$2^{3^7} \equiv$	120211220022222222_3	$3^{2^7} \equiv$	100111101000000001_2

Table 3: Power powers of n in $2^{3^n} (mod\, 3^{18})$ base 3 and $3^{2^n}\ (mod\, 2^{18})$ base 2

and 7 being coprime with 10 and that $a^{\varphi(b^m)} \equiv 1\ (mod\, b^m)$. But what happens with the double powers of 2 which has a divisor of 10? Simply choose a base like 3 to work in that is coprime to 2.

With Table 2 and Table 3 to reference, we propose the idea of a universal digital independence where under iterated exponentiation, that the digits of least significance become independent of the digits of greater significance. Consider the triangle of zeros in Table 3 begining with d_2 in 3^{2^1}, then $d_3 d_2$ in 3^{2^2}.

The process of iterated exponentiation acts to pump the entropy out of the least significant digits until each successive exponentiation freezes the next unfrozen least significant digit.

$$(b+a)^{b^n} = \sum_{k=0}^{b^n} b^k a^{b^n-k} \frac{b^n!}{(b^n-k)!k!}$$

$$= b^0 a^{b^n} \frac{b^n!}{(b^n)!0!} + b^1 a^{b^n-1} \frac{b^n!}{(b^n-1)!1!} + b^2 a^{b^n-2} \frac{b^n!}{(b^n-2)!2!} + \cdots + b^{b^n-2} a^2 \frac{b^n!}{(b^n-2)!2!} + b^{b^n-1} a^1 \frac{b^n!}{(b^n-1)!1!} + b^{b^n} a^0$$

$$= a^{b^n} + a^{b^n-1} b^{n+1} + \frac{1}{2!} a^{b^n-2} b^{n+2} (b^n-1) + \cdots + \frac{1}{2!} a^{n+2} b^{b^n-2} (b^n-1) + a^{n+1} b^{b^n-1} + b^{b^n}$$

$$= a^{b^n} + b^n \left[a^{b^n-1} b + \frac{1}{2!} a^{b^n-2} b^2 (b^n-1) + \cdots + \frac{1}{2!} a^{n+2} b^{b^n-n-2} (b^n-1) + a^{n+1} b^{b^n-n-1} + b^{b^n-n} \right]$$

$$(b+a)^{b^n} \equiv a^{b^n} (\ mod\ b^n)$$

Let $a = 1$ then

Theorem $(b+1)^{b^n} \equiv 1(\ mod\ b^n)$.

Proof:

$$(b+1)^{b^n} \equiv 1^{b^n} \equiv 1(\ mod\ b^n)$$

□

Let $a = -1$ then

$$(b-1)^{b^n} \equiv -1^{b^n} (\ mod\ b^n)$$

If b is even, $b \equiv 0(\ mod\ 2)$ then $(b-1)^{b^n} \equiv -1^{b^n} \equiv 1(\ mod\ b^n)$
If b is odd, $b \equiv 1(\ mod\ 2)$ then $(b-1)^{b^n} \equiv -1^{b^n} \equiv -1(\ mod\ b^n)$
Theorem If b is even then $(b-1)^{b^n} \equiv 1(\ mod\ b^n)$.
Proof:

$$(b+1)^{b^n} \equiv -1^{b^n} \equiv -1^{2^n k^n} \equiv 1(\ mod\ b^n)$$

□

Theorem If b is odd then $(b-1)^{b^n} \equiv -1(\ mod\ b^n)$.
Proof:

$$(b+1)^{b^n} \equiv -1^{b^n} \equiv -1(\ mod\ b^n)$$

□

Definition *Digital Independence*, For a positive integer $a \geq 2$, a coprime base $b = a-1$ and $\alpha < \beta$, the digit d_α is independent if for every digit of greater significance d_β, $a^{d_\beta} \equiv \pm 1 (mod\ b^\alpha)$. Then $a^{d_\beta} a^{d_\alpha} \equiv \pm a^{d_\alpha} (mod\ b^\alpha)$

Theorem For positive integer $a > 2$, the digit d_n becomes independent on the n^{th} exponential iteration.

Proof: Set $b = a-1$, then the base b is coprime with a. Prove by induction,

Basis Step: $n = 1$ Then $a^{b^1} \equiv 1\ (mod\ b)$. Then d_1 is independent on the first iteration since d_1 is associated with b^0 in ${}^1a = \cdots d_2 \cdot b^1 + d_1 \cdot b^0$.

Induction Step: Let $n = k$ and assume $d_{n-1} \cdots d_1$ is independent and that d_n becomes independent. Prove that on the $n = k+1$ iteration that $d_n \cdots d_1$ remains independent and that d_{k+1} becomes independent. Consider that

$$a^{\varphi(b^{k+1})} = a^{b^{k+1-1}\varphi(b)} = (a^{\varphi(b)})^{b^{k+1-1}} \equiv 1\ (mod\ b^{k+1})$$

Therefore $d_k \cdots d_1$ remains independent and d_{k+1} becomes independent since d_{k+1} is associated with b^k in $d_{k+1} \cdot b^k + d_k \cdot b^{k-1} + \cdots + d_2 \cdot b^1 + d_1 \cdot b^0$. This completes the induction step and finishes the proof. □

Theorem If digits recur in one base they recur in all bases.

Proof: Consider a positive integer n with recuring least significant digits in base α. For any other arbitrary base β, a third base $\alpha \cdot \beta$ exists that must also have recuring digits. But since the base $\alpha \cdot \beta$ has recuring digits, the base β also has recuring digits. Since the choice of b was arbitrary, we have proven that if digits recur in one base then they recur in all bases. □

2 Tetrates of n

Theorem The digits of the tetrates of a number ${}^n a$ in a coprime base b gain one recuring digit with every exponential iteration for $n \geq 2$.

Proof: For positive integer $a > 2$, the digit d_n becomes independent on the n^{th} exponential iteration. □

The standard use of base ten results in most positive integers not being coprime. For $k \geq 6$, the tetrates of even numbers end in

$$(2 \cdot n)^{(10^k)} \equiv 7109376 \ (mod\ 10^7)$$

while the tetrates of numbers divisable by 5 end with

$$(5 \cdot n)^{(10^k)} \equiv 2890625 \ (mod\ 10^7)$$

otherwise

$$n^{10^k} \equiv 1 (mod\ 10^7)$$

$$7109376 + 2890625 = 10000001 \equiv 1 \ (mod\ 10^7)$$

$$7109376 \cdot 2890625 = 20550540000000 \equiv 0 \ (mod\ 10^7)$$

Recurring digits in tetration where $n \geq 10$.

${}^n1 \equiv$	1	${}^n11 \equiv$	2666611
${}^n2 \equiv$	2948736	${}^n12 \equiv$	4012416
${}^n3 \equiv$	4195387	${}^n13 \equiv$	5045053
${}^n4 \equiv$	1728896	${}^n14 \equiv$	7502336
${}^n5 \equiv$	8203125	${}^n15 \equiv$	859375
${}^n6 \equiv$	7238656	${}^n16 \equiv$	415616
${}^n7 \equiv$	5172343	${}^n17 \equiv$	85777
${}^n8 \equiv$	5225856	${}^n18 \equiv$	4315776
${}^n9 \equiv$	2745289	${}^n19 \equiv$	9963179

Table 4: Tetration ($mod\ 10^7$), where $n \geq 10$

3 Ackermann Function

Consider that the tetrates of an integer are a subset of its powers, and the pentates of an integer are a subset of its tetrates. By induction, if reccuring least significant digits occur in tetrates, then they occur in all of the higher arithmatic operators of the Ackermann function.

The Ackermann function can be written using Knuth arrow notation defined as $a \to n \to 1 = a^n$ for exponentiation, $a \to n \to 2 = {}^na$ for tetration, $a \to n \to 3$ for pentation and so forth.

Consider the previous table. Since $n \geq 10$, the expression

$${}^n2 \equiv 2948736 \ (mod\ 10^7)2$$

could be replaced with the following for $k \geq 2$,

$$2 \to n \to k \equiv 2948736 \ (mod\ 10^7).$$

Theorem The recurring digits in pentation $a \to n \to 3$ grow approximately as $a \to (n-1) \to 3$.

Proof: Note that $a \to n \to 3 = a \to (a \to (n-1) \to 3) \to 2$, but $a \to k \to 2$ has approximately k recurring digits, therefore $a \to n \to 3$ has approximately $a \to (n-1) \to 3$ digits. $\square$